Turbulent kinetic energy and a possible hierarchy of length scales in a generalization of the Navier-Stokes α theory

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We present a continuum-mechanical formulation and generalization of the Navier-Stokes α theory based on a general framework for fluid-dynamical theories with gradient dependencies. Our flow equation involves two additional problem-dependent length scales α and β. The first of these scales enters the theory through the internal kinetic energy, per unit mass, $\alpha^2|\mathbf{D}|^2$, where $\mathbf{D}$ is the symmetric part of the gradient of the filtered velocity. The remaining scale is associated with a dissipative hyperstress which depends linearly on the gradient of the filtered vorticity. When α and β are equal, our flow equation reduces to the Navier-Stokes α equation. In contrast to the original derivation of the Navier-Stokes α equation, which relies on Lagrangian averaging, our formulation delivers boundary conditions. For a confined flow, our boundary conditions involve an additional length scale ℓ characteristic of the eddies found near walls. Based on a comparison with direct numerical simulations for fully developed turbulent flow in a rectangular channel of height $2h$, we find that $\alpha/\beta \sim \mathrm{Re}^{0.470}$ and $\ell/h \sim \mathrm{Re}^{-0.772}$, where Re is the Reynolds number. The first result, which arises as a consequence of identifying the internal kinetic energy with the turbulent kinetic energy, indicates that the choice $\alpha=\beta$ required to reduce our flow equation to the Navier-Stokes α equation is likely to be problematic. The second result evinces the classical scaling relation $\eta/L \sim \mathrm{Re}^{-3/4}$ for the ratio of the Kolmogorov microscale η to the integral length scale L. The numerical data also suggests that $\ell \leqslant \beta$. We are therefore led to conjecture a tentative hierarchy, $\ell \leqslant \beta < \alpha$, involving the three length scales entering our theory.

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I. BACKGROUND

The Lagrangian averaged Navier-Stokes α model for (statistically homogeneous and isotropic) turbulent flow yields a governing equation for the filtered velocity $\mathbf{v}$ that can be expressed in the form

$$\rho\dot{\mathbf{v}} = -\operatorname{grad} p + \mu(1-\alpha^2\Delta)\Delta\mathbf{v} + 2\rho\alpha^2 \operatorname{div} \overset{\circ}{\mathbf{D}}; \tag{1}$$

(1) is commonly referred to as the Navier-Stokes α equation. In this equation: $\mathbf{v}$ is subject to the incompressibility constraint

$$\operatorname{div} \mathbf{v} = 0; \tag{2}$$

$\dot{\mathbf{v}} = \partial\mathbf{v}/\partial t + (\operatorname{grad}\mathbf{v})\mathbf{v}$ is the material time derivative of $\mathbf{v}$; p is the filtered pressure; Δ is the Laplace operator;

$$\mathbf{D} = \frac{1}{2}[\operatorname{grad}\mathbf{v} + (\operatorname{grad}\mathbf{v})^{\mathsf{T}}] \tag{3}$$

is the filtered stretch rate;

$$\overset{\circ}{\mathbf{D}} = \dot{\mathbf{D}} + \mathbf{D}\mathbf{W} - \mathbf{W}\mathbf{D}, \tag{4}$$

is the corotational rate of $\mathbf{D}$, with

$$\mathbf{W} = \frac{1}{2}[\operatorname{grad}\mathbf{v} - (\operatorname{grad}\mathbf{v})^{\mathsf{T}}]$$

the filtered spin. The Lagrangian averaged Euler equation, which is (1) with $\mu=0$, was originally derived by Holm, Marsden, and Ratiu [1,2]. Subsequently, Chen, Foias, Holm, Olson, Titi, and Wynne [3–5] added the viscous term to the Lagrangian averaged Euler equation, giving (1).

Aside from the density ρ and the shear viscosity μ, the flow equation (1) involves an additional parameter $\alpha>0$ carrying dimensions of length. Within the framework of Lagrangian averaging, α is the statistical correlation length of the excursions taken by a fluid particle away from its phase-averaged trajectory. More intuitively, α is often interpreted as the characteristic linear dimension of the smallest eddies that the model is capable of resolving. Like equations arising from Reynolds averaging, the Navier-Stokes α equation provides an approximate model that resolves motions only above some critical scale, while relying on filtering to approximate effects at smaller scales. A synopsis of properties and advantages of the Navier-Stokes α equation is provided by Holm, Jeffrey, Kurien, Livescu, Taylor, and Wingate [6].

The structure of (1) is formally suggestive of a conservation law expressing the balance of linear momentum, and one might ask whether there is a complete continuum mechanical framework in which the Navier-Stokes α equation is embedded along with suitable boundary conditions. Based on experience with theories for structured media [7], the presence of a term involving the fourth-order spatial gradient of the velocity indicates that any such framework should involve a hyperstress in addition to the classical Cauchy stress. Within the context of turbulence theory, a hyperstress might be viewed as providing a means to account for interactions across disparate length scales.

II. PRINCIPLE OF VIRTUAL POWER

To see the need for an additional hyperstress assume an inertial frame, neglect noninertial body forces, and note first that the weak form of the classical momentum balance

$$\operatorname{div}\mathbf{T}+\mathbf{b}=\mathbf{0}, \tag{5}$$

with inertial force

$$\mathbf{b}=-\rho\dot{\mathbf{v}} \tag{6}$$

treated for convenience as a body force, has the form

$$\underbrace{\int_{\partial R}\mathbf{t}(\mathbf{n})\cdot\boldsymbol{\phi}a+\int_R\mathbf{b}\cdot\boldsymbol{\phi}dv}_{\mathcal{W}_{\text{ext}}(R,\boldsymbol{\phi})}=\underbrace{\int_R\mathbf{T}:\operatorname{grad}\boldsymbol{\phi}dv}_{\mathcal{W}_{\text{int}}(R,\boldsymbol{\phi})}, \tag{7}$$

with

$$\mathbf{t_n}=\mathbf{Tn} \tag{8}$$

the classical surface traction of Cauchy. Granted smoothness (7) holds for all virtual velocities (i.e., test fields) $\boldsymbol{\phi}$ and all control volumes R if and only if the balance (5) is satisfied at all points in the fluid and the traction condition (8) is satisfied—for any choice of the unit vector $\mathbf{n}$—at all points in the fluid. Moreover, the requirement of frame-indifference applied to (7) yields the symmetry of the stress $\mathbf{T}$.

When $\boldsymbol{\phi}$ represents the velocity $\mathbf{v}$, the weak balance (7) is a physical balance

$$\underbrace{\int_{\partial R}\mathbf{t_n}\cdot\mathbf{v}da+\int_R\mathbf{b}\cdot\mathbf{v}dv}_{\mathcal{W}_{\text{ext}}(R)}=\underbrace{\int_R\mathbf{T}:\operatorname{grad}\mathbf{v}\,dv}_{\mathcal{W}_{\text{int}}(R)} \tag{9}$$

between

(i) the external power $\mathcal{W}_{\text{ext}}(R)$, which represents power expended on R by tractions acting on ∂R and power expended by the inertial force $\mathbf{b}$;

(ii) the internal power $\mathcal{W}_{\text{int}}(R)$, the integrand of which represents the classical stress power $\mathbf{T}:\operatorname{grad}\mathbf{v}$ expended within R by the stress field $\mathbf{T}$.

Here and in what follows, we write $\mathcal{W}_{\text{ext}}(R)$ for the external power associated with an actual flow and $\mathcal{W}_{\text{ext}}(R,\boldsymbol{\phi})$ for the (virtual) external power associated with a virtual velocity field $\boldsymbol{\phi}$.

The balance (7) represents a nonstandard version of the classical principle of virtual power, as formulated by Gurtin [8]. This nonstandard form has been generalized by Fried and Gurtin [7] to develop a gradient theory for liquid flows at small length scales and, when combined with suitable constitutive relations, results in a partial differential equation slightly more general than (1) but with the term involving the corotational rate of $\mathbf{D}$ removed. Conventional versions of this principle are formulated for the fluid region as a whole rather than for control volumes and as such generally involve particular boundary conditions. Here the principle of virtual power is used instead as a basic tool in determining the structure of the tractions and of the local force balances. As such, conditions on the external boundary play a role no different from those on the boundary of any control volume. Basic to this view is the premise, central to all of continuum mechanics, that any basic law for the body should hold also for all subregions of the body. On more pragmatic grounds, the nonstandard formulation allows for the derivation of the associated angular momentum balance.

To capture the internal power associated with the formation of eddies during turbulent flow we generalize the classical theory by including in the internal power a term linear in the vorticity gradient

$$\operatorname{grad}\boldsymbol{\omega}=\operatorname{grad}\operatorname{curl}\mathbf{v}.$$

Specifically, we introduce a second-order tensor-valued hyperstress $\mathbf{G}$ via an internal power expenditure of the form $\mathbf{G}:\operatorname{grad}\boldsymbol{\omega}$ and rewrite the power expended within R in the form

$$\mathcal{W}_{\text{int}}(R)=\int_R(\mathbf{T}:\operatorname{grad}\mathbf{v}+\mathbf{G}:\operatorname{grad}\boldsymbol{\omega})dv. \tag{10}$$

In conjunction with the internal power expenditure (10), we introduce a corresponding external power expenditure

$$\mathcal{W}_{\text{ext}}(R)=\int_{\mathcal{S}}\left(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{v}+\mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\right)da+\int_R\mathbf{b}\cdot\mathbf{v}dv, \tag{11}$$

in which $\mathbf{t}_{\mathcal{S}}$ and $\mathbf{m}_{\mathcal{S}}$ represent tractions on the bounding surface $\mathcal{S}=\partial R$ of R, while $\mathbf{b}$ represents the inertial body force (6). Here the term

$$\mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}, \tag{12}$$

which is not present in classical theories, is needed to balance the effects of the internal-power term $\mathbf{G}:\operatorname{grad}\boldsymbol{\omega}$, which involves the second gradient of $\mathbf{v}$.

The principle of virtual power replaces $\mathbf{v}$ by $\boldsymbol{\phi}$ and (hence) $\boldsymbol{\omega}$ by $\operatorname{curl}\boldsymbol{\phi}$ and is based on the requirement that the virtual expenditures of internal and external power be equal,

$$\underbrace{\int_{\mathcal{S}}\left(\mathbf{t}_{\mathcal{S}}\cdot\boldsymbol{\phi}+\mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\boldsymbol{\phi}}{\partial n}\right)da+\int_R\mathbf{b}\cdot\boldsymbol{\phi}dv}_{\mathcal{W}_{\text{ext}}(R,\boldsymbol{\phi})}=\underbrace{\int_R(\mathbf{T}:\operatorname{grad}\boldsymbol{\phi}+\mathbf{G}:\operatorname{grad}\operatorname{curl}\boldsymbol{\phi})dv}_{\mathcal{W}_{\text{int}}(R,\boldsymbol{\phi})} \tag{13}$$

for all control volumes R and any choice of the virtual velocity field $\boldsymbol{\phi}$.

III. LOCAL BALANCE LAW FOR LINEAR MOMENTUM: TRACTION CONDITIONS

Consequences of the principle of virtual power and the additional stipulation that the internal power expenditure be frame indifferent are that

(i) The classical macroscopic balance $\rho\dot{\mathbf{v}}=\operatorname{div}\mathbf{T}$ must be replaced by the balance

$$\rho\dot{\mathbf{v}} = \operatorname{div}\mathbf{T} + \operatorname{curl}\operatorname{div}\mathbf{G}, \tag{14}$$

with $\mathbf{T}$ symmetric as in the classical theory, viz.,

$$\mathbf{T}^{\mathsf{T}} = \mathbf{T}. \tag{15}$$

(ii) Cauchy's classical condition $\mathbf{t_n}=\mathbf{Tn}$ for the traction across a surface $\mathcal{S}$ with unit normal $\mathbf{n}$ must be replaced by the conditions

$$\mathbf{t}_{\mathcal{S}} = \mathbf{Tn} + \operatorname{div}_{\mathcal{S}}(\mathbf{Gn}\times) + \mathbf{n}\times(\operatorname{div}\mathbf{G} - 2K\,\mathbf{Gn}),$$

$$\mathbf{m}_{\mathcal{S}} = \mathbf{n}\times\mathbf{Gn}, \tag{16}$$

where $\operatorname{div}_{\mathcal{S}}$ is the divergence operator on $\mathcal{S}$, given a vector $\mathbf{a}$, $\mathbf{a}\times$ is the second-order tensor defined so that $(\mathbf{a}\times)\mathbf{b}=\mathbf{a}\times\mathbf{b}$ for all vectors $\mathbf{b}$, and $K=-\frac{1}{2}\operatorname{div}_{\mathcal{S}}\mathbf{n}$ is the mean curvature of $\mathcal{S}$. Thus, interestingly, the traction $\mathbf{t}_{\mathcal{S}}$ depends on the mean curvature; in fact, the term $\operatorname{div}_{\mathcal{S}}(\mathbf{Gn}\times)$ results in a dependence on the full curvature tensor $\mathbf{K}=-\operatorname{grad}_{\mathcal{S}}\mathbf{n}$.

Being independent of constitutive assumptions, these results apply to fluids and solids alike.

Within the framework of finite deformations of an elastic solid with couple stress, the balance (14) was first derived by the Cosserats [9]; see, also, Toupin [10,11], Mindlin and Tiersten [12], and Green and Naghdi [13]. The traction conditions (16) are special cases of traction conditions derived variationally by Toupin [10,11] for the boundary of the elastic solid.

The balance (14) and the traction conditions (16) are special cases of Eqs. (5.11) and (5.12) of Fried and Gurtin [7], whose theory replaces curl $\boldsymbol{\omega}$ in the internal power with the full second gradient $\operatorname{grad}^2\mathbf{v}$ and $\mathbf{G}$ by an analogous third-order hyperstress; see, also, Bluestein and Green [14], who discuss second-gradient fluids based on the multipolar theory of second-gradient materials due to Green and Rivlin [15]. The multipolar theory results in redundant boundary conditions, which Bluestein and Green [14] reduce using *ad hoc* arguments.

IV. ENERGETICS

We restrict attention to a purely mechanical theory based on the requirement that the temporal increase in free energy of an arbitrary region $\mathcal{R}(t)$ that convects with the body be less than or equal to the power expended on that region. Precisely, letting ψ denote the specific free energy, this requirement takes the form of a free-energy imbalance

$$\frac{d}{dt}\int_{\mathcal{R}(t)}\rho\psi dv \le \mathcal{W}_{\rm ext}[\mathcal{R}(t)], \tag{17}$$

where $\mathcal{W}_{\rm ext}$ is as defined in (11). The imbalance (17) is consistent with standard continuum thermodynamics based on balance of energy and an entropy imbalance (the Clausius-Duhem inequality), when that imbalance is restricted to isothermal processes.

Balance of mass implies that

$$\frac{d}{dt}\int_{\mathcal{R}(t)}\rho\psi dv = \int_{\mathcal{R}(t)}\rho\dot{\psi} dv;$$

since $\mathcal{W}_{\rm ext}[\mathcal{R}(t)]=\mathcal{W}_{\rm int}[\mathcal{R}(t)]$, we may therefore use the expression (10) for the internal power $\mathcal{W}_{\rm int}[\mathcal{R}(t)]$ in conjunction with the requirement (15) that $\mathbf{T}$ be symmetric, to localize (17); the result is the local free-energy imbalance

$$\rho\dot{\psi} - \mathbf{T}{:}\mathbf{D} - \mathbf{G}{:}\operatorname{grad}\boldsymbol{\omega} \le 0, \tag{18}$$

where $\mathbf{D}$ as introduced in (3) is the stretching.

V. SIMPLE CONSTITUTIVE EQUATIONS

We assume that the fluid is incompressible, so that

$$\rho = \text{constant} \quad \text{and} \quad \operatorname{div}\mathbf{v} = \operatorname{tr}\mathbf{D} = 0. \tag{19}$$

Without loss in generality, we may then suppose that

$$\mathbf{T} = \mathbf{S} - p\mathbf{1}, \quad \text{with } \operatorname{tr}\mathbf{S} = 0, \tag{20}$$

where the *pressure* p is a constitutively indeterminate field that does not affect the internal power (10); the field $\mathbf{S}$ represents the extra stress. Then, by the second equation in (19),

$$\mathbf{T}{:}\mathbf{D} = \mathbf{S}{:}\mathbf{D} \tag{21}$$

and the local free-energy imbalance (18) reduces to

$$\mathbf{S}{:}\mathbf{D} + \mathbf{G}{:}\operatorname{grad}\boldsymbol{\omega} - \rho\dot{\psi} \ge 0. \tag{22}$$

Guided by the presence of the term involving the corotational rate of the stretching tensor $\mathbf{D}$ in the Navier-Stokes α equation (1), we suppose that the specific free energy ψ and the extra stress $\mathbf{S}$ are given by constitutive equations of the form

$$\psi = \alpha^2|\mathbf{D}|^2 \quad \text{and} \quad \mathbf{S} = 2\mu\mathbf{D} + 2\rho\alpha^2\overset{\circ}{\mathbf{D}}, \tag{23}$$

with $\alpha \ge 0$ and μ constant. These choices are familiar from the theory of Rivlin-Ericksen fluids; see, also, Rivlin and Ericksen [16], Sec. 119 of Truesdell and Noll [17], and Dunn and Fosdick [18]. For turbulent flow, $\psi=\alpha^2|\mathbf{D}|^2$ might best be viewed as a (specific) internal kinetic energy which accounts for the dispersive transfer of energy between eddies of different scales. Hereafter, we use this terminology.

Futher, we assume that the hyperstress $\mathbf{G}$ depends linearly on the vorticity gradient. It then follows that

$$\mathbf{G} = \mu\beta^2[\operatorname{grad}\boldsymbol{\omega} + \gamma(\operatorname{grad}\boldsymbol{\omega})^{\mathsf{T}}], \tag{24}$$

with $\beta \ge 0$ and γ constant. With the choices (23) and (24), the dissipation inequality (22) holds if and only if

$$\mu \ge 0 \quad \text{and } |\gamma| \le 1. \tag{25}$$

Whereas μ is the conventional shear viscosity, the constitutive parameters α and β carry dimensions of length. Whereas α is related to the internal kinetic energy ψ and, therefore, to the nondissipative contribution to the extra stress $\mathbf{S}$, β is associated with the wholly dissipative hyperstress $\mathbf{G}$. To ensure that ψ has a strict minimum when $\mathbf{D}=\mathbf{0}$, we assume that

$$\alpha > 0. \tag{26}$$

Further, to ensure that the hyperstress is nontrivial when grad $\boldsymbol{\omega} \neq \mathbf{0}$, we assume that

$$\beta > 0. \tag{27}$$

When discussing turbulence, it is conventional to divide the range of eddy scales into integral, inertial, and dissipative subranges (aside from the classical contributions of Richardson [19] and Kolmogorov [20–22], see Pope [23]). The integral scales are the largest and are associated with external driving forces. The dissipative scales are the smallest and are associated with the conversion of kinetic energy into heat. The intermediate, inertial, scales are commonly thought to be dissipationless. It seems reasonable to expect that the energetic length α should represent a characteristic average of the eddy scales within the inertial subrange whereas β should represent a characteristic average of the eddy scales within the dissipative subrange, in which case α and β would obey

$$\beta < \alpha. \tag{28}$$

Using (23) and (24) in (14) and bearing in mind that the moduli μ, α, β, and γ are assumed to be constant, we arrive at the flow equation

$$\rho\dot{\mathbf{v}} = -\,\mathrm{grad}\,p + \mu(1-\beta^2\Delta)\Delta\mathbf{v} + 2\rho\alpha^2\,\mathrm{div}\,\mathring{\mathbf{D}}, \tag{29}$$

which, for the particular choice $\beta = \alpha$ specializes to the Navier-Stokes α equation (1). In view of the foregoing discussion, this particular choice embodies a questionable assumption concerning the relationship between the scales of inertial and dissipative eddies. We refer to (29) as the Navier-Stokes $\alpha\beta$ equation.

The parameter γ, which is dimensionless, does not enter the flow equation (29). However, as is clear from the second equation in (16) and (24), γ would generally be present in any boundary condition in which the hypertraction is prescribed. [In this regard, consider the condition (35).]

By assuming that the internal kinetic energy ψ, extra stress $\mathbf{S}$, and hyperstress $\mathbf{G}$ are as given in (23) and (24), we are motivated primarily by the desire to obtain the simplest possible framework encompassing the Navier-Stokes α equation (1). Precedent for (23) can be traced to Rivlin [24,25], who showed that the laminar flow of a non-Newtonian fluid in a duct with an elliptical cross section is necessarily accompanied by secondary flow in the cross-sectional plane and, thereby, noted the analogy between the turbulent flow of a Newtonian fluid and the laminar flow of a non-Newtonian fluid. An overview of history, advantages, and limitations of nonlinear eddy-viscosity models based on Rivlin-Ericksen fluids of second and higher grade is provided by Gatski [26]. A discussion of the connection between the Navier-Stokes α model and non-Newtonian rheological models is provided by Foias, Holm and Titi [27].

VI. BOUNDARY CONDITIONS

We develop counterparts of the classical notions of a free surface and a fixed surface without slip. Our results hinge on rewriting the external power expenditure (11) for the entire fluid body B and focusing on that portion of that expenditure associated with tractions. In this regard, we derive boundary force and moment balances

$$\mathbf{t}_{\mathcal{S}} = \mathbf{t}^{\mathrm{env}}_{\partial B} + 2\sigma K\mathbf{n} \quad \text{and} \quad \mathbf{m}_{\mathcal{S}} = \mathbf{m}^{\mathrm{env}}_{\partial B} \tag{30}$$

giving the tractions $\mathbf{t}_{\mathcal{S}}$ and $\mathbf{m}_{\mathcal{S}}$ in terms of their environmental counterparts $\mathbf{t}^{\mathrm{env}}_{\partial B}$ and $\mathbf{m}^{\mathrm{env}}_{\partial B}$, and use these balances to express the power expended by tractions in the form

$$\mathcal{P}(\partial B) \stackrel{\mathrm{def}}{=} \int_{\partial B}\left((\mathbf{t}^{\mathrm{env}}_{\partial B} + 2\sigma K\mathbf{n})\cdot\mathbf{v} + \mathbf{m}^{\mathrm{env}}_{\partial B}\cdot\mathbf{P}\frac{\partial\mathbf{v}}{\partial n}\right)da, \tag{31}$$

where $\mathbf{P} = \mathbf{1} - \mathbf{n}\otimes\mathbf{n}$. We assume that the mean curvature K of—and the surface tension σ at—the boundary ∂B are known; (31) then suggests that a class of reasonable boundary conditions, at each point of ∂B, consists of

(i) a prescription of $\mathbf{t}^{\mathrm{env}}_{\partial B}$ or $\mathbf{v}$, or a relation between $\mathbf{t}^{\mathrm{env}}_{\partial B}$ and $\mathbf{v}$; and

(ii) a prescription of $\mathbf{m}^{\mathrm{env}}_{\partial B}$ or $\mathbf{P}\partial\mathbf{v}/\partial n$, or a relation between these quantities.

Consistent with this observation, we consider specific boundary conditions in which a portion $\mathcal{S}_{\mathrm{free}}$ of ∂B is a free surface and the remainder $\mathcal{S}_{\mathrm{nslp}}$ is a fixed surface without slip. On $\mathcal{S}_{\mathrm{free}}$, the environmental tractions $\mathbf{t}^{\mathrm{env}}_{\partial B}$ and $\mathbf{m}^{\mathrm{env}}_{\partial B}$ vanish and the classical condition $\mathbf{Tn} = \sigma K\mathbf{n}$ is replaced by the conditions

$$\mathbf{Tn} + \mathrm{div}_{\mathcal{S}}(\mathbf{Gn}\times) = \sigma K\mathbf{n} \quad \text{and } \mathbf{n}\times\mathbf{Gn} = \mathbf{0}. \tag{32}$$

To describe the conditions on $\mathcal{S}_{\mathrm{nslp}}$, we first note that, if $\mathbf{v} = \mathbf{0}$ on $\mathcal{S}_{\mathrm{nslp}}$, then

$$\mathbf{P}\frac{\partial\mathbf{v}}{\partial n} = \boldsymbol{\omega}\times\mathbf{n} \tag{33}$$

with $\boldsymbol{\omega} = \mathrm{curl}\,\mathbf{v}$ the vorticity.

Based on this identity, we take, as boundary condition on $\mathcal{S}_{\mathrm{nslp}}$, the classical no-slip condition

$$\mathbf{v} = \mathbf{0} \tag{34}$$

supplemented by a condition of the form

$$\mathbf{n}\times\mathbf{Gn} = \mathbf{m}^{\mathrm{env}}_{\partial B} \tag{35}$$

with

$$\mathbf{m}^{\mathrm{env}}_{\partial B} = \mu\ell\,\boldsymbol{\omega}\times\mathbf{n}, \tag{36}$$

where ℓ carries dimensions of length. We refer to (36) as the wall-eddy condition and to ℓ as the wall-eddy length. In the wall-eddy condition, $\mathbf{m}^{\mathrm{env}}_{\partial B}$ represents the hypertraction induced by the formation of eddies at a fixed surface without slip. The wall-eddy condition requires that this hypertraction be parallel to the tangential vorticity. Hence, $\mathbf{m}^{\mathrm{env}}_{\partial B}$ arises in response to the shedding of vortices at the boundary.

By (33) and (36), the power expenditure by $\mathbf{m}^{\mathrm{env}}_{\partial B}$ in (31) has the form

$$\mathbf{m}^{\mathrm{env}}_{\partial B}\cdot(\boldsymbol{\omega}\times\mathbf{n}) = \mu\ell|\boldsymbol{\omega}\times\mathbf{n}|^2. \tag{37}$$

The quantity $\boldsymbol{\omega}|^2$ is known as the *enstrophy*; the field $|\boldsymbol{\omega}\times\mathbf{n}|^2$ might therefore be viewed as a tangential enstrophy

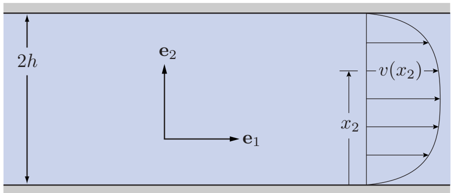

FIG. 1. (Color online) Schematic for the problem of flow in a channel of gap $2h$. The coordinates in the directions downstream and out of the plane are x_1 and x_3.

associated with the shedding of vortices at the wall.

Note that (35) and (36) combine with (24) to yield the wall-eddy condition in the form

$$\mathbf{n} \times \{\beta^2[\operatorname{grad} \boldsymbol{\omega} + (\operatorname{grad} \boldsymbol{\omega})^{\mathsf{T}}]\mathbf{n} + \ell \boldsymbol{\omega}\} = \mathbf{0}. \tag{38}$$

VII. FLOW IN A RECTANGULAR CHANNEL

We now consider the problem of a steady, turbulent flow through an infinite, rectangular channel formed by two parallel walls separated by a gap $2h$ (Fig. 1). We suppose that channel walls are fixed and without slip in the sense that the no-slip and wall-eddy conditions (34) and (36) hold. This simple model problem allows us to investigate the effects of the parameters α, β, and ℓ and to make comparisons with numerical results.

A. Explicit solution of the channel problem

Employing the notation of Fig. 1, we assume that the velocity $\mathbf{v}$ has the form

$$\mathbf{v}(\mathbf{x}) = v(x_2)\mathbf{e}_1; \tag{39}$$

$\mathbf{v}$ is therefore consistent with the constraint (2) of incompressibility and obeys $\dot{\mathbf{v}}=0$. In view of (39), the Navier-Stokes $\alpha\beta$ equation (29) gives

$$\mu(v - \beta^2 v'')'' = \frac{\partial p}{\partial x_1},$$

$$2\rho\alpha^2 v' v'' = \frac{\partial p}{\partial x_2},$$

$$0 = \frac{\partial p}{\partial x_3}, \tag{40}$$

while the no-slip and wall-eddy conditions (34) and (38) give

$$v(0) = 0,$$

$$v(2h) = 0,$$

$$\beta^2 v''(0) = -\ell v'(0),$$

$$\beta^2 v''(2h) = \ell v'(2h). \tag{41}$$

In (40) and (41) and what follows a prime is used to denote differentiation with respect to the spanwise coordinate x_2.

Since v depends only on x_2, (40) implies that

$$p(x_1,x_2) = -Ax_1 + \rho\alpha^2|v'(x_2)|^2, \tag{42}$$

with $A=$constant. We assume, without loss of generality, that the pressure decreases with increasing x_2. It then follows that

$$A > 0. \tag{43}$$

Further, in view of (40)–(42), v can be expressed as

$$v(x_2) = \frac{Ah^2}{2\mu}\left[1 - \left(1 - \frac{x_2}{h}\right)^2 + \frac{2B}{\dfrac{h}{\beta}}\left(1 - \frac{\cosh \dfrac{h}{\beta}\left(1 - \dfrac{x_2}{h}\right)}{\cosh \dfrac{h}{\beta}}\right)\right], \tag{44}$$

with

$$B = \frac{\dfrac{\ell}{\beta} - \dfrac{\beta}{h}}{1 - \dfrac{\ell}{\beta}\tanh \dfrac{h}{\beta}}. \tag{45}$$

To ensure that (44) is nonsingular, the wall-eddy length ℓ is assumed consistent with

$$\frac{\ell}{\beta}\tanh\frac{h}{\beta} \neq 1. \tag{46}$$

B. Behavior at the wall

Experiments and DNS simulations of channel flow show that, for suitably normalized laminar and turbulent velocity profiles, the slopes of the turbulent profiles at the channel walls have magnitudes greater than their laminar counterparts (see, for example, Pope [23]). Consistent with this observation, we normalize v by its maximum value to yield

$$V(x_2) = \frac{v(x_2)}{v(h)}. \tag{47}$$

For comparison, we introduce

$$V_c(x_2) = 1 - \left(1 - \frac{x_2}{h}\right)^2, \tag{48}$$

which is the analogous normalization of the laminar solution to the plane channel problem. Then, $V'(0) > V_c'(0)$ if and only if

$$B = \frac{\dfrac{\ell}{\beta} - \dfrac{\beta}{h}}{1 - \dfrac{\ell}{\beta}\tanh\dfrac{h}{\beta}} > 0. \tag{49}$$

Since $\beta > 0$ and $h > 0$ it follows that v as defined by (44) captures the observed features of turbulent channel flow only if the wall-eddy length obeys

$$\ell > \frac{\beta^2}{h} > 0. \tag{50}$$

TABLE I. Values of h/β, ℓ/β, and θ determined by fitting v^+ to the DNS data of Kim, Moin, and Moser [28] and Moser, Kim, and Mansour [29] for the nominal values $\mathrm{Re}_\tau=180$, $\mathrm{Re}_\tau=395$, and $\mathrm{Re}_\tau=590$ of the friction Reynolds number.

Re_τ	h/β	ℓ/β	θ
180	16.6	0.957	0.0583
395	34.8	0.974	0.0336
590	48.1	0.980	0.0239

C. Comparison with numerical data for the velocity profile

Assuming that wall-eddy length ℓ obeys (50), we now compare the solution v to the problem for channel flow to the mean downstream velocity for turbulent channel flow as predicted by the direct numerical simulations (DNS) of Kim, Moin, and Moser [28] and Moser, Kim, and Mansour [29].

To facilitate comparisons, we employ standard definitions for the friction velocity v_τ, friction Reynolds number Re_τ, and the viscous length y^+:

$$v_\tau = \sqrt{\frac{\tau_w}{\rho}}, \quad \mathrm{Re}_\tau = \frac{\rho h v_\tau}{\mu}, \quad y^+ = \frac{\mathrm{Re}_\tau}{h} x_2, \tag{51}$$

with $\tau_w > 0$ being the wall shear stress. (Throughout this section, we employ the terminology and notation of Pope [23].) In addition, we introduce a dimensionless velocity v^+ via

$$v^+(y^+) = \frac{1}{v_\tau} v\left(\frac{h}{\mathrm{Re}_\tau} y^+\right), \tag{52}$$

with v as given by (44). In view of (51) and (52),

$$v^+(y^+) = \frac{\mathrm{Re}_\tau \theta}{2}\left[1 - \left(1 - \frac{y^+}{\mathrm{Re}_\tau}\right)^2 + \frac{2B}{\frac{h}{\beta}}\left(1 - \frac{\cosh \frac{h}{\beta}\left(1 - \frac{y^+}{\mathrm{Re}_\tau}\right)}{\cosh \frac{h}{\beta}}\right)\right], \tag{53}$$

where θ is defined by the pressure drop A, the half-channel height h, and the wall shear stress τ_w by

$$\theta = \frac{Ah}{\tau_w}. \tag{54}$$

We use the nonlinear least-squares method to fit v^+ as defined by (52) to the average downstream velocity profile determined by the DNS simulations of Kim, Moin, and Moser [28] and Moser, Kim, and Mansour [29] for the nominal values $\mathrm{Re}_\tau=180$, $\mathrm{Re}_\tau=395$, and $\mathrm{Re}_\tau=590$ of the friction Reynolds number. The values of the parameters h/β, ℓ/β [note from (45) that B as defined in (45) depends on both h/β and ℓ/β], and θ determined by these fits are listed in Table I and plots of v^+ corresponding to these fits are shown, along with the DNS data, in Fig. 2. These data show that the ratios ℓ/h and β/h are on the order of 10^{-2}. These ratios

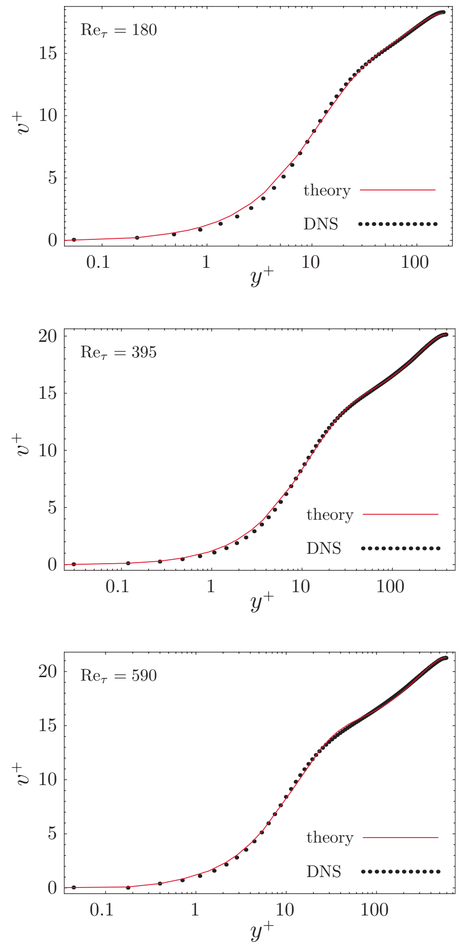

FIG. 2. (Color online) Comparison of the dimensionless velocity v^+ with the downstream velocity determined by the DNS simulations of Kim, Moin, and Moser [28] and Moser, Kim, and Mansour [29] for the nominal values $\mathrm{Re}_\tau=180$, $\mathrm{Re}_\tau=395$, and $\mathrm{Re}_\tau=590$ of the friction Reynolds number.

therefore correspond to dimensionless lengths in the lower half of the buffer layer.

The second and third columns of Table I combine to yield data relating ℓ/h to Re_τ. A power-law fit then shows that $\ell/h \sim \mathrm{Re}_\tau^{-0.882}$ (Fig. 3). If we invoke the empirical resistance law $\mathrm{Re}_\tau \sim \mathrm{Re}^{7/8}$ of Blasius [30], we find that

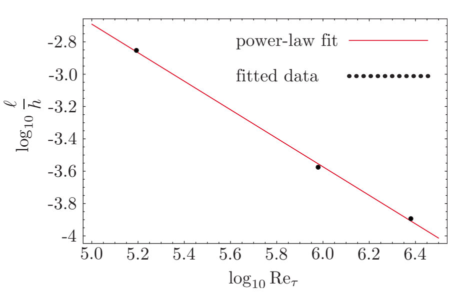

FIG. 3. (Color online) Plot of $\log_{10}(\ell/h)$ versus $\log_{10}\mathrm{Re}_\tau$, as determined by the fitted data in Table I. The straight line shows a power-law fit of the form $\ell/h \sim \mathrm{Re}_\tau^{-0.882}$, with $\chi^2 = 1.11 \times 10^{-4}$.

$$\frac{\ell}{h} \sim \mathrm{Re}^{-0.772}, \tag{55}$$

where Re denotes the Reynolds number. If we identify h with the integral length L and ℓ with the Kolmogorov length η the result (55) is then strikingly close to the classical scaling relation [20–23]

$$\frac{\eta}{L} \sim \mathrm{Re}^{-3/4} \tag{56}$$

for the ratio of the largest to smallest length scales present in a turbulent flow.

Conversely, supposing that $\ell/h \sim \mathrm{Re}^{-3/4}$ and using the relation $\ell/h \sim \mathrm{Re}_\tau^{-0.882}$, we find that $\mathrm{Re}_\tau \sim \mathrm{Re}^{0.850}$ in close agreement with the Blasius [30] resistance law.

Another interesting feature of the data in Table I is that it suggests that ℓ increases monotonically with Re_τ and should most likely obey the limit

$$\lim_{\mathrm{Re}_\tau \to \infty} \ell = \beta. \tag{57}$$

Granted (57), the wall-eddy length ℓ would be less than or equal to the dissipative length scale β,

$$\ell \leqslant \beta. \tag{58}$$

This is consistent with the view that the distribution of eddy scales represented near the boundary of a flow domain should be dominated by the smallest scales present in the flow [31].

Granted (58), it then follows from (50) that the wall-eddy length must obey

$$\ell < h. \tag{59}$$

This inequality is certainly consistent with the scaling relation (55).

D. Comparison with numerical data for the turbulent kinetic-energy profile

Due to the simple nature of channel flow, the velocity field as determined by (39) and (44) is independent of the energetic length scale α. Information concerning that scale can nevertheless be obtained by identifying the internal kinetic energy $\psi = \alpha^2 |\mathbf{D}|^2$ with the turbulent kinetic energy. With this identification, we will find that agreement with data generated by DNS simulations requires that the energetic length scale α be substantially larger than the dissipative length scale β.

By (39) and (44), $\psi = \alpha^2 |\mathbf{D}|^2 = \alpha^2 |u'|^2/2$. Thus, introducing the dimensionless internal kinetic energy $k^+ = 2\psi/v_\tau^2$ and using the nondimensionalization (51) and (52), we find that

$$k^+(y^+) = \frac{\alpha^2 \mathrm{Re}_\tau^2}{2h^2} \left| \frac{dv^+(y^+)}{dy^+} \right|^2. \tag{60}$$

Taking the previously obtained values of h/β, ℓ/β, and θ, we use the nonlinear least-squares method to fit k^+ to the turbulent kinetic-energy (i.e., one-half the trace of the Reynolds stress tensor) determined by the DNS simulations of Kim, Moin, and Moser [28] and Moser, Kim, and Mansour [29]. The values of α/h determined by these fits are listed in Table II and plots of k^+ corresponding to these fits are shown, along with the DNS data, in Fig. 4. Quite interestingly, the values of α/h coincide approximately with the upper bound of the log-law region.

TABLE II. Values of α/h determined by fitting k^+ the turbulent kinetic energy determined by the DNS simulations of Kim, Moin, and Moser [28] and Moser, Kim, and Mansour [29] for the nominal values $\mathrm{Re}_\tau=180$, $\mathrm{Re}_\tau=395$, and $\mathrm{Re}_\tau=590$ of the friction Reynolds number.

Re_τ	α/h
180	0.359
395	0.258
590	0.237

Although the overall trend of the fits agrees with the data, their detailed features show some deviations. In particular, the fitted peak values of the turbulent kinetic energy occur

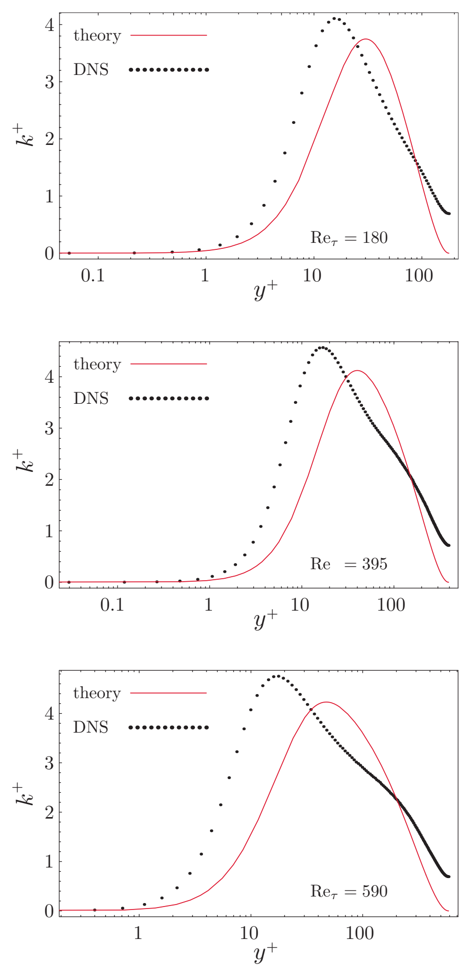

FIG. 4. (Color online) Comparison of the dimensionless internal kinetic energy k^+ with the turbulent kinetic energy determined by the DNS simulations of Kim, Moin, and Moser [28] and Moser, Kim, and Mansour [29] for the nominal values $\mathrm{Re}_\tau=180$, $\mathrm{Re}_\tau=395$, and $\mathrm{Re}_\tau=590$ of the friction Reynolds number.

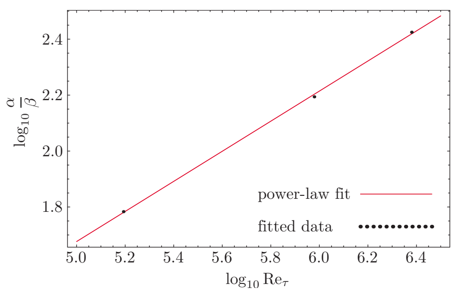

FIG. 5. (Color online) Plot of $\log_{10}(\alpha/\beta)$ versus $\log \mathrm{Re}_\tau$, as determined by the fitted data in Tables I and II. The straight line shows a power-law fit of the form $\alpha/\beta \sim \mathrm{Re}_\tau^{0.538}$, with $\chi^2=2.43\times10^{-4}$.

too far from the channel walls and are too low. As a consequence, the turbulent kinetic energy is too low in most of the buffer layer and too high in the log-law region. Also, the fitted turbulent kinetic energy vanishes, incorrectly, at the center of the channel. These shortcomings might be attributed to the one-dimensional nature of the analytical model. Bearing in mind that the DNS simulations are three dimensional, the fits are unexpectedly good.

Combining the second columns of Tables I and II, we arrive at data relating α/β and Re_τ. A power-law fit then shows that $\alpha/\beta \sim \mathrm{Re}_\tau^{0.538}$ (Fig. 5). If we again invoke the Blasius [30] resistance law, we find that

$$\frac{\alpha}{\beta} \sim \mathrm{Re}^{0.471}. \tag{61}$$

For turbulent flow ($\mathrm{Re}\gg 1$), this result is consistent with our previously expressed view that the energetic length scale α should exceed the dissipative length scale β.

The importance of allowing the energetic and dissipative length scales to differ is underscored by the foregoing results. For the Navier-Stokes α model, $\alpha=\beta$ is determined by fitting the velocity profile. Since the values of β/h are less than those of α/h by two orders of magnitude, the corresponding peak values of the dimensionless internal kinetic-energy for the Navier-Stokes α model must be lower by four orders of magnitude than those obtained for the Navier-Stokes $\alpha\beta$ model. In this sense, it would be unphysical to identify $\psi=\alpha|\mathbf{D}|^2$ with the turbulent kinetic energy in the Navier-Stokes α model.

The above considerations demonstrate that the length scales α, β, and ℓ should be viewed as problem-dependent flow parameters rather than constitutive moduli that characterize different fluids.

VIII. FREE-ENERGY IMBALANCE AT THE WALL

Recently Fried and Gurtin [7] provided a general discussion of the use of a free-energy imbalance for a boundary pillbox to develop constitutive relations describing the interaction of the fluid and its environment. We now sketch the corresponding analysis, but only as it applies to the no-slip boundary conditions (34) and (36). Thus let $\mathcal{S}$ denote a fixed (i.e., time independent), subsurface of $\mathcal{S}_{\mathrm{nslp}}$. We find it useful

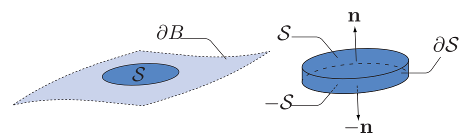

FIG. 6. (Color online) Pillbox corresponding to a subsurface $\mathcal{S}$ of the boundary ∂B of the region B of space occupied by the body. Only a portion of ∂B is depicted. Whereas $\mathbf{n}$ is oriented into the environment, $-\mathbf{n}$ is oriented into the fluid.

to view $\mathcal{S}$ as a boundary pillbox of infinitesimal thickness involving:

(i) a surface $\mathcal{S}$ with unit normal $\mathbf{n}$; $\mathcal{S}$ is viewed as lying in the environment at the interface of the fluid and the environment;

(ii) a surface $-\mathcal{S}$ with unit normal $-\mathbf{n}$; $-\mathcal{S}$ is viewed as lying in the fluid adjacent to the interface with the environment;

(iii) a lateral face represented by $\partial\mathcal{S}$.

See Fig. 6.

Let ψ^x denote the excess free energy, measured per unit area, of the fluid at the surface $\mathcal{S}_{\mathrm{nslp}}$, so that

$$\int_{\mathcal{S}} \psi^x da \tag{62}$$

represents the net free energy of the pillbox.

Consider next the power expended on the pillbox surface $-\mathcal{S}$ by the fluid. With this in mind, note first that, by (33) and (34), the power expenditure (31) due to hypertractions reduces to

$$\mathcal{P}(\partial B) = \int_{\partial B} \mathbf{m}_{\partial B}^{\mathrm{env}} \cdot (\boldsymbol{\omega}\times\mathbf{n})da. \tag{63}$$

This relation (63) represents the net power expended on the fluid at the boundary. Thus, since $\mathbf{m}_{-\mathcal{S}}=\mathbf{m}_{\mathcal{S}}=\mathbf{m}_{\partial B}^{\mathrm{env}}$, the power expended by the fluid on the pillbox surface $-\mathcal{S}$ has the form

$$\int_{\mathcal{S}} \mathbf{m}_{-\mathcal{S}}\cdot[\boldsymbol{\omega}\times(-\mathbf{n})]da = -\int_{\mathcal{S}} \mathbf{m}_{\partial B}^{\mathrm{env}}\cdot(\boldsymbol{\omega}\times\mathbf{n})da. \tag{64}$$

We assume that the power expended by the environment on the pillbox surface $\mathcal{S}$ vanishes and hence that the environment is passive. Further, we neglect hyperstresses within the fluid-environment interface. Hence there is no expenditure of power on the lateral face of the pillbox. Thus, if we parallel the development in bulk with the requirement that the temporal increase in free energy of $\mathcal{S}$ be less than or equal to the power expended on $\mathcal{S}$, then we arrive at the free-energy imbalance [cf. (17)],

$$\underbrace{\frac{d}{dt}\int_{\mathcal{S}} \psi^x da}_{\text{free-energy rate}} - \underbrace{\int_{\mathcal{S}} [-\mathbf{m}_{\partial B}^{\mathrm{env}}\cdot(\boldsymbol{\omega}\times\mathbf{n})]da}_{\text{power expenditure}} \le 0. \tag{65}$$

Since $\mathcal{S}$ is fixed, we may interchange the operations of integration and time differentiation in (65); thus, since $\mathcal{S}$ is arbitrary, if we appeal to (36), we are led to the inequality

$$\mu\ell|\boldsymbol{\omega}\times\mathbf{n}|^2 \leqslant -\dot{\psi}^x. \tag{66}$$

In our discussion of channel flow in Sec. VII B we note that the velocity field as characterized by our theory captures the observed features of turbulent channel flow only *if* the wall-eddy length obeys

$$\ell > 0 \tag{67}$$

[cf. (50)]. This conclusion is underlined by the fact that for channel flow the theory with $\ell>0$ agrees well with the DNS simulations.

On the other hand, for channel flow $\dot{\psi}^x \equiv 0$, because the flow is steady and $\mathbf{v}\equiv\mathbf{0}$ at the wall. The free-energy imbalance (66) therefore becomes

$$\mu\ell|\boldsymbol{\omega}\times\mathbf{n}|^2 \leqslant 0 \tag{68}$$

and is violated when $\ell>0$. This observation would seem to indicate a conceptual error in the free-energy imbalance (65). In fact there is such an error.

Indeed, the field $\mathbf{v}$ is not the actual fluid velocity $\mathbf{v}_{\rm act}$, but instead a filtered velocity representing an average of $\mathbf{v}_{\rm act}$; consequently, $\boldsymbol{\omega}$ represents a filtered vorticity. Thus—in terms more suggestive than precise—the left-hand side of the free-energy imbalance (65) represents, for a pillbox $\mathcal{S}$, a difference of the form

$$\frac{d}{dt}\{\text{filtered free energy of }\mathcal{S}\} - \{\text{power expended on }\mathcal{S} \text{ over the filtered tangential vorticity}\} \tag{69}$$

and hence does not account for power and energy associated with the actual motion of the fluid at the small scales (i.e., those scales which have been filtered and are not included). While it is to be expected that a free-energy imbalance should be satisfied in any flow, laminar or turbulent, it would seem unreasonable to require that filtered variables obey a free-energy imbalance.

In principle, we may account for the power expenditures and energy rates at the small scales via a "supply term"

$$-\{\text{effective supply of energy to } \mathcal{S} \text{ due to behavior at the filtered scales}\}; \tag{70}$$

in this manner we are led to consider a generalization of (65) in the form

$$\frac{d}{dt}\int_{\mathcal{S}}\psi^x da - \int_{\mathcal{S}}[-\mathbf{m}^{\rm env}_{\partial B}\cdot(\boldsymbol{\omega}\times\mathbf{n})]da - \{\text{effective supply of energy to } \mathcal{S} \text{ due to behavior at the filtered scales}\} \leqslant 0. \tag{71}$$

Our theory is, of course, too coarse to determine a specific form of the effective energy supply. Based on this observation we do not consider (65) to be a viable free-energy imbalance and consider the theory as complete without (65).

Finally, if we assume that the flow at the wall is dissipationless, then the inequality in (71) becomes an equality and, granted that the effective supply of energy has a local form measured per unit area on $\mathcal{S}_{\rm nslp}$, we can trivially compute the local effective supply for channel flow,

$$\{\text{local effective supply of energy of } \mathcal{S} \text{ due to behavior at the filtered scales}\} = \mu\ell|\boldsymbol{\omega}\times\mathbf{n}|^2.$$

IX. SUMMARY

The generalization of the Navier-Stokes α model discussed here involves the Navier-Stokes $\alpha\beta$ equation

$$\rho\dot{\mathbf{v}} = -\operatorname{grad} p + \mu(1-\beta^2\Delta)\Delta\mathbf{v} + 2\rho\alpha^2 \operatorname{div}\mathring{\mathbf{D}} \tag{72}$$

for the filtered velocity $\mathbf{v}$ and, for a confined flow, the no-slip boundary condition

$$\mathbf{v} = \mathbf{0} \tag{73}$$

and the wall-eddy condition

$$\mathbf{n}\times\{\beta^2[\operatorname{grad}\boldsymbol{\omega} + (\operatorname{grad}\boldsymbol{\omega})^{\rm T}]\mathbf{n} + \ell\boldsymbol{\omega}\} = \mathbf{0}, \tag{74}$$

where $\mathbf{n}$ denotes the outward unit normal to the boundary and $\boldsymbol{\omega}=\operatorname{curl}\mathbf{v}$ is the filtered vorticity.

The Navier-Stokes $\alpha\beta$ equation and wall-eddy condition involve three problem-dependent length scales α, β, and ℓ, with α and β being of energetic and dissipative origin, respectively, and the wall-eddy length ℓ being associated with the characteristic scale of eddies at the boundary. Conventional views concerning the distribution of eddy scales in the inertial and dissipative subranges suggest that $\beta<\alpha$. Our study of flow in a rectangular channel upholds this expectation, showing that, based on an identification between the internal kinetic-energy $\psi=\alpha^2|\mathbf{D}|^2$ with the turbulent kinetic energy,

$$\frac{\alpha}{\beta} \sim \mathrm{Re}^{0.471}. \tag{75}$$

Even for low Reynolds number turbulent flows, α must therefore be substantially larger than β. Furthermore, consideration of the velocity profile indicates that

$$\frac{\ell}{h} \sim \mathrm{Re}^{-0.772} \tag{76}$$

and that

$$\ell \leqslant \beta. \tag{77}$$

We are therefore led to conjecture a tentative hierarchy,

$$\ell \leqslant \beta < \alpha, \tag{78}$$

involving the three length scales.

A complete derivation of the Navier-Stokes $\alpha\beta$ equation and wall-eddy condition, including discussion of the mechanical and thermodynamic foundations of our theory, is provided by Fried and Gurtin [32].

ACKNOWLEDGMENTS

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